N64-25001

CODE NONE cat 08

JOURNAL OF GEOPHYSICAL RESEARCH VOL. 69, NO. 3 FEBRUARY 1, 1964

Ion Sheath Effects near Antennas Radiating within the Ionosphere

H. A. WHALE[1]

Goddard Space Flight Center, Greenbelt, Maryland

Abstract. A theoretical treatment of the electron displacement in the vicinity of a linear cylindrical antenna immersed in the ionosphere has been developed which explains the surprisingly thick ion sheaths that have been observed experimentally when large RF voltages are applied to the antenna. The force that displaces the electrons is obtained from numerical solutions to the nonlinear differential equation describing their motion, and the results are found to be consistent with the observations.

author

INTRODUCTION

Early observations of changes in the impedance of an electrically short antenna in the ionosphere [*Jackson*, 1952; *Jackson and Pickar*, 1957] led to the method of determining the dielectric constant (and hence the electron density) of the ionosphere from measurements of the capacity of a linear antenna [*Jackson and Kane*, 1959, 1960; *Kane et al.*, 1962].

It is well known that an ion sheath a few centimeters thick normally forms spontaneously around a body immersed in the ionosphere, and it has been shown by *Kane et al.* [1962] that, if the RF voltage applied to the antenna is small (less than 2 volts at 7.75 Mc/s), the sheath thickness observed agrees approximately with that predicted by *Jastrow and Pearse* [1957]. However, when large RF voltages are applied to the antenna, e.g. 200 volts at 7.75 Mc/s, the ion sheath that forms appears to be very much larger. The redistribution of the electrons under the influence of RF fields is investigated in the present paper, and it is found that a relatively simple analysis yields values of sheath diameter in good agreement with the experimental results.

ION SHEATH ARISING FROM VEHICLE POTENTIAL

The Jastrow and Pearse treatment of the sheath problem proceeds in two steps. First, the vehicle potential is computed from a knowledge of the electron temperature and density of the ionosphere. Second, this calculated potential is used to derive the sheath thickness. In practice, thermal- and photo-emission effects change the actual potential of the vehicle in a way that is difficult to predict, so that an actual measurement of the potential is necessary. Such measurements that have been made (passim) indicate that the vehicle potential is generally in the range 0 to -1 volt with respect to the ambient medium [*Kane et al.*, 1962; *Bourdeau et al.*, 1960]. The size of the sheath that forms around a long cylindrical surface for different values of electron density and vehicle potential can then be found from the following simplified calculations:

Assume that the ion sheath is a sharply bounded cylindrical region around the antenna and that there are no electrons within this region. The transition region from sheath to plasma will actually be smoothed out because of the thermal motions of the electrons, but we will assume that there is an equivalent sharp boundary or sheath edge. This sheath edge is assumed to be at zero (plasma) potential.

The capacity of the antenna to this surface is

$$C = \frac{2\pi\epsilon_0}{\log (R/R_0)} \text{ farads/m} \qquad (1)$$

where

R = sheath radius.
R_0 = antenna radius.
ϵ_0 = $(36\pi \times 10^9)^{-1}$ farad/m.

Thus the charge Q on the antenna is

$$Q = 2\pi\epsilon_0 V/\log (R/R_0) \text{ coulombs/m}$$

where V = antenna potential in volts, and the field E_1 at the sheath boundary is

$$E_1 = V/\{R \log (R/R_0)\} \text{ volts/m} \qquad (2)$$

[1] Permanent address: University of Auckland, New Zealand.

Inasmuch as the effect of the sheath is to shield the ambient medium from the effects of any charge on the antenna, we can also equate this field from the antenna with that arising from the electron density deficiency. The polarization field arising from the electron deficiency is

$$E_2 = \frac{(R^2 - R_0^2)Ne}{2R\epsilon_0} \text{ volts/m} \qquad (3)$$

where N = ambient density in electrons/cubic meter, and e = electronic charge in coulombs.

Equating (2) and (3) to obtain the equilibrium condition, we can find the value of R from

$$2\epsilon_0 V = Ne(R^2 - R_0^2) \log (R/R_0)$$

The solution to this equation is plotted in a convenient form in Figure 1, from which the ratio R/R_0 is readily obtained for different values of V/N (V in volts and N in electrons per cubic meter) for various values of R_0 (meters). The usual simplifying assumption of negligible magnetic field effects is made in the above treatment.

The sheath size can be increased if the antenna is biased negatively with respect to the vehicle. The influence of the ionosphere on the antenna reactance is thus decreased, and this effect is helpful when it is desirable to maintain an approximately constant antenna impedance.

If the antenna voltage varies sufficiently slowly, the sheath thickness will also vary, the maximum frequency at which the sheath can follow the voltage changes being close to the electron plasma frequency [*Rose and Clark*, 1961]. It does not follow that the electrons remain stationary when the applied frequency is greater than the plasma frequency, but rather

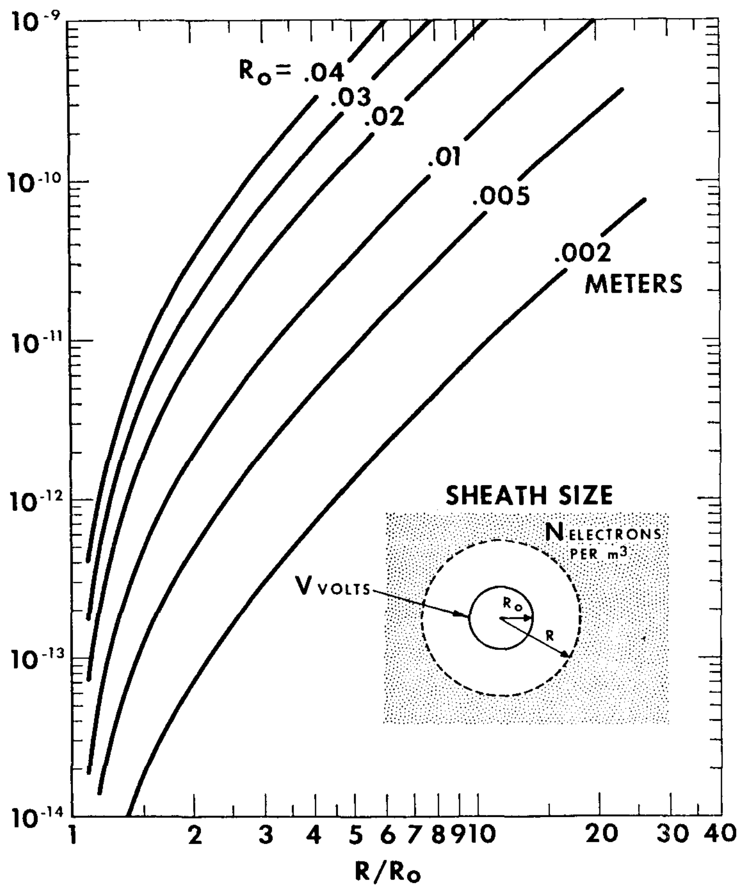

Fig. 1. Effective sheath size as a function of antenna potential (in volts) and electron density (per cubic meter). Inset gives nomenclature.

that they can not rearrange themselves quickly enough to shield out the applied RF field. In the following treatment we are concerned with motions of the electrons at frequencies above the plasma frequency, so that it is their time-averaged positions that are affected by any sheath effects.

Motion of a Single Electron

At distances from a long cylindrical antenna that are small compared with the antenna length, the field is essentially perpendicular to the antenna axis. If effects arising from any magnetic field or from collisions or other losses are neglected, the electron motions will be along the field lines. The strength of the cylindrically radial field (i.e. perpendicular to the antenna axis) is inversely proportional to the distance from the axis. The motion of a single electron is thus described by the nonlinear differential equation

$$m\ddot{r} = (eE_0/r)\cos\omega t \tag{5}$$

where

e, m = charge and mass of the electron.
r = distance of electron from antenna axis in meters.
E_0 = peak field strength at a distance of 1 meter from the axis, and is found from
$E_0 = V_0 C/2\pi\epsilon_0$ volts/m.

where

$V_0\cos\omega t$ = antenna voltage at any point measured with respect to the medium.
C = antenna capacity per unit length.
$\epsilon_0 = (36\pi \times 10^9)^{-1}$ farad/m.
$\omega = (2\pi) \times$ (operating frequency).

Equation 5 can be solved by approximate methods or by numerical computation of electron trajectories. It is found that, although the exciting voltage is sinusoidal, the fact that the electron experiences forces that are no longer sinusoidal (because of the inverse distance term) introduces an average force which accelerates an individual electron toward the lower field-strength region. The same phenomenon occurs when there are many electrons present, provided that the frequency is above the plasma frequency, as is shown in the next section.

Motion of Electrons in an Ionized Medium

When the antenna is immersed in an ionized medium, all the electrons will oscillate in the RF field and will also tend to move away from the antenna, but, since this motion produces a polarization of the medium, an equilibrium condition is attained. Thus, with an RF field applied, a sheath appears, the sheath radius increasing with the field strength. The sheath edge, together with the electrons in the medium, vibrates at the applied frequency. In order to solve this problem we require steady-state solutions to a modified form of (5) which include the terms arising from the restoring forces. The deficit of negative charge in a cylindrical space of radius r free of electrons around the antenna is

$$Q = \pi r^2 Ne/\mathrm{m}$$

where it has been assumed that the space occupied by the antenna itself is negligible compared with the space that is free of electrons. The surface field at the sheath edge arising from this charge is

$$F = Q/2\pi r\epsilon_0 = Ner/2\epsilon_0 \text{ volts/m} \tag{6}$$

which is the same as equation 3 with $R_o = 0$.

The equation of motion for an electron at the sheath edge thus becomes

$$m\ddot{r} = -Fe + \frac{eE_0}{r}\cos\omega t \tag{7}$$

$$= -\frac{Ne^2 r}{2\epsilon_0} + \frac{eE_0}{r}\cos\omega t$$

Thus

$$\ddot{r} = -\frac{\omega_p^2 r}{2} + \frac{eE_0}{mr}\cos\omega t \tag{8}$$

where

$$\omega_p = (2\pi) \times (\text{plasma frequency})$$
$$= (Ne^2/m\epsilon_0)^{1/2}$$

Equation 8 can be written in the following form, where the differentiations are with respect to a new variable $\tau(\tau = \omega t)$ and where the notation y represents $r(\tau)$ and $\ddot{y}$ represents $d^2r/d\tau^2$:

$$\ddot{y} = -\frac{\omega_p^2 y}{2\omega^2} + \frac{eE_0}{m\omega^2 y}\cos\tau$$

where τ is in radians, leading to the simplified expression

$$\ddot{y} = -Ay + (B/y) \cos \tau \tag{9}$$

where

$$A = X/2(X = \omega_p^2/\omega^2)$$

and

$$B = eE_0/m\omega^2$$

= amplitude of oscillation of an isolated single electron in a uniform field of strength E_0

Equation 9 can be regarded as specifying the trajectory of an electron under specified values of B, which depends on the antenna voltage and the frequency, and A, which depends on the electron density and frequency for particular initial conditions. In general, the solution will describe an electron vibrating at about the applied frequency and with its center of motion oscillating at a frequency close to $\omega_p/2\sqrt{2}\pi$. This oscillation may, in some cases, carry the electron on to the antenna or completely out of the field. Our interest is in the steady-state solution, i.e. the condition in which each cycle of the oscillation at the applied frequency is a repeat of previous cycles. In order to find these solutions, values of B and the initial radial distance y (when $\dot{y} = 0$) were chosen, and then A was varied in a systematic manner until a repetitive (steady-state) solution was obtained. The physical interpretation is as follows. The term Ay in (9) can be written

$$Ay = A\bar{y} + A\delta y \tag{10}$$

where $\bar{y}$ = time average value of y, and δy = instantaneous displacement of the electron from $\bar{y}$.

The term $A\delta y$ in (10) then represents the effect of the polarization in returning the electron to its stable position at $\bar{y}$, and thus $A\bar{y}$ is a measure of the steady polarization required to balance the mechanical force tending to move the electron away from the antenna. For a given B, the steady-state trajectories for a series of initial y's can be found, each yielding a value for A and thus X (since $A = X/2$). The magnitude of $\bar{y}$ can be found from the calculated trajectory, and a plot can be prepared in which $\bar{y}$ is shown as a function of X for various values of the parameter B. A series of such solutions is drawn in Figure 2.

The calculated value of $\bar{y}$ can be taken as the effective average sheath radius, since electrons closer to the antenna will experience, on the average, larger repelling forces and will tend to move out to this position. The more distant electrons in the medium will also oscillate in the RF field and will experience two forces tending to displace them from their mean positions. These two forces are: (1) The field that arises from the electron deficiency near the antenna and that varies as $1/r$, tending to move the electrons toward the antenna; (2) The average force that tends to move the electrons into the region of low RF field strength.

The resulting motion of an electron at $z (z > y)$ is then described by the differential equation

$$\ddot{z} = C - A(y^2/z) + (B/z) \cos \tau$$

where y satisfies

$$\ddot{y} = -Ay + (B/y) \cos \tau$$

and C is a constant that can be adjusted so that a stable solution is obtained. Solution of these equations yields small positive values of C, indicating that the electrons in the medium drift toward the antenna. This drift is to be expected, since the field from the electron-deficient region near the antenna falls off as $1/r$, while the force arising from the movement of the electrons falls off as $1/r^4$ approximately (as in the approximate analysis in the next section leading to equation 15).

This effect may lead to some local increase in the electron density just outside the sheath region, i.e. a moderate electron sheath forms outside the positive ion sheath. It is probable that this boundary region, assumed to be a step function for the purposes of the above analysis, will merge into a relatively smooth transition region.

Approximate Solutions

Approximate solutions to equation 9 can be obtained by using certain simplifying assumptions. Assuming that the motion of the electrons is sinusoidal, we write

$$y = \bar{y} + a \cos \tau \tag{11}$$

Then

$$\ddot{y} = -a \cos \tau$$

Substituting in (9), neglecting the second harmonic terms that appear, and equating the

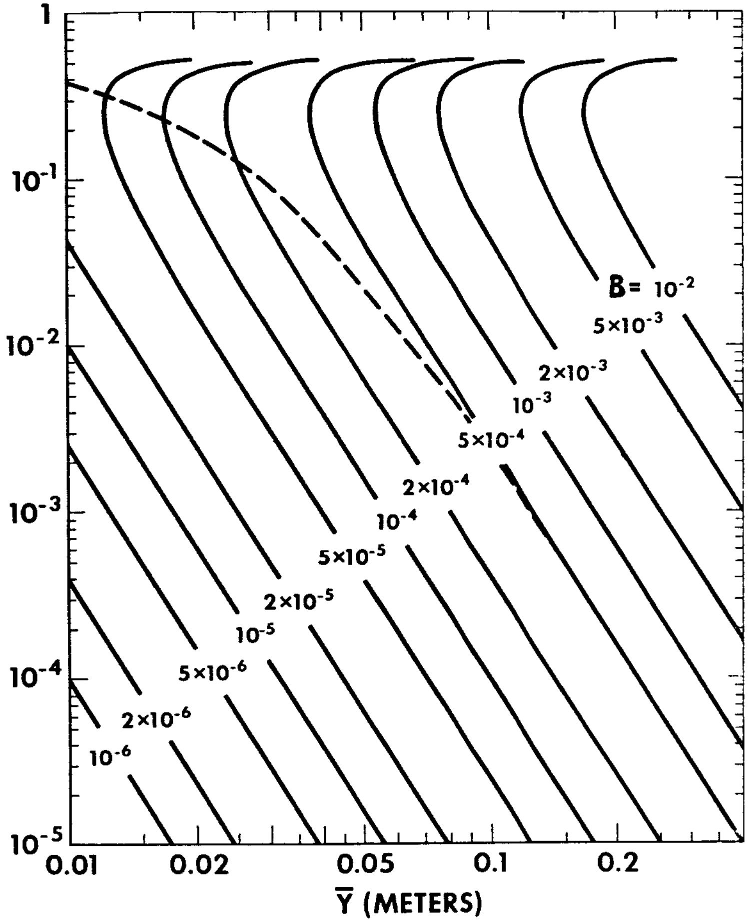

Fig. 2. Effective sheath radius ($\bar{Y}$) as a function of $X(=\omega_p^2/\omega^2)$ for various values of the parameter B. The dashed line shows the closest distance from the antenna for electrons vibrating about the mean position $\bar{Y}$ for $B = 5 \times 10^{-4}$.

steady and the oscillatory terms in the resulting equation, we obtain

$$\begin{aligned} a &= [B/(2A - 1)]\bar{y} \\ &= -[B/(1 - X)]\bar{y} \end{aligned} \tag{12}$$

and

$$\begin{aligned} a^2 &= 2A\bar{y}^2/(1 - A) \\ &= 2X\bar{y}^2/(2 - X) \end{aligned} \tag{13}$$

Eliminating a, we find

$$\begin{aligned} y^{-4} &= \frac{(2 - X)B^2}{2X(1 - X)^2} \\ &\approx B^2/X \quad \text{for small } X \end{aligned} \tag{14}$$

and thus

$$A = X/2 \approx B^2/2\bar{y}^4 \tag{15}$$

Further harmonic terms could be added by assuming a series for y in (11). This process leads to a more accurate solution but becomes progressively more complicated. It is therefore much simpler to use the computed curves than to attempt to increase the accuracy of these approximate solutions by including more harmonic terms.

Discussion

It is seen from (15) that for small values of X the mean distance $\bar{y}$ of the electrons from the antenna axis decreases as X increases, in agreement with the computed curves in Figure 2. With large X, $\bar{y}$ increases again. Some resonance effect would be expected from (8), since, if the driving voltage E_0 is removed, the equation becomes

$$\ddot{r} = -(\omega_p^2 r/2)$$
$$= \text{constant} - (\omega_p^2/2)\,\delta r$$

where δr = excursion of the sheath edge from its average position, with the oscillatory solution

$$\omega^2 = \omega_p^2/2$$

Note that, for the particular geometry that has been studied here, the simplified approach indicates that this resonance is not exactly at the plasma frequency, but at

$$\omega = \omega_p/\sqrt{2}$$

If the sheath is very small, it is not permissible to take $r_0 \ll r$, as was assumed in (6). With $r_0 \approx r$, (8) becomes (with $E_0 = 0$)

$$\ddot{r} = -\omega_p^2(r^2 - r_0^2)/2r$$
$$\approx -\omega_p^2(r - r_0)$$
$$= \text{constant} - \omega_p^2\,\delta r$$

In this case the resonance occurs more nearly at the plasma frequency. The nonlinear motion of the electrons complicates the resonance phenomenon, and a more detailed investigation than is attempted in this particular paper is required.

In interpreting these results it must be remembered that generally there is also a dc sheath formed around any antenna immersed in the ionosphere. This condition arises from the charge acquired by the vehicle from electron capture. The dc sheath thickness (in the absence of any RF fields applied to the antenna) is commonly of the order of a centimeter or two; hence, although this dc effect may be negligible compared with the effects arising from the RF field at high voltages, it will be the controlling influence when the RF field is very small.

Numerical Example

Let us consider a typical situation, taking the following conditions: antenna radius 0.01 m, antenna capacity 16 $\mu\mu$f/m, frequency 7.75 Mc/s, peak RF voltage 200 volts; then

$$E_0 = V_0 C/2\pi\epsilon_0$$
$$= 57.6 \text{ v/m},$$

so that

$$B = eE_0/m\omega^2$$
$$= E_0/224.3f^2 \text{ m } (f \text{ in Mc/s})$$
$$= 4.3 \times 10^{-3} \text{ m}$$

From Figure 2 we see that as X increases, the mean distance of the inner layer of electrons from the antenna axis decreases from 0.22 m at $X = 0.01$ to 0.125 m at $X = 0.1$ with a minimum of 0.105 m at about $X = 0.3$.

The minimum distance from the antenna axis attained by the electrons nearest to the antenna during their oscillations is also known. It is the initial value of y (for $\dot{y} = 0$) used in the computations of the solution of the differential equation. This distance is shown, for example, as the dashed curve in Figure 2 for the conditions of $B = 5 \times 10^{-4}$. It is seen in this case that, neglecting the effects of any dc bias on the antenna, electrons will strike the antenna (of radius 0.01 m) when the electron density becomes greater than the value corresponding to $X = 0.4$.

Experimental Observations

The equation from which Figure 2 was derived is

$$\ddot{y} = -Ay + (B/y)\cos\tau$$

If the scale of y is changed by a factor n, we can write

$$n\ddot{y} = -Any + n^2B/ny\cos\tau$$

or

$$\ddot{Y} = -AY + (B'/Y)\cos\tau$$

where

$$Y = ny \qquad B' = n^2B$$

It is thus possible to draw all the curves in Figure 2 as a single curve with suitable normalizing factors. This type of curve is shown in Figure 3, where the computed results are normalized to $B = 10^{-2}$ and where only the portion of the curve for $X > 0.1$ is shown.

There are also a few experimental results normalized to $B = 10^{-2}$, plotted in Figure 3. These values were obtained during two rocket firings, NN 3.08F [*Jackson and Kane*, 1959] and NASA 4.07 [*Kane et al.*, 1962].

In the NN 3.08F experiment the antenna impedance was obtained from measurements of the

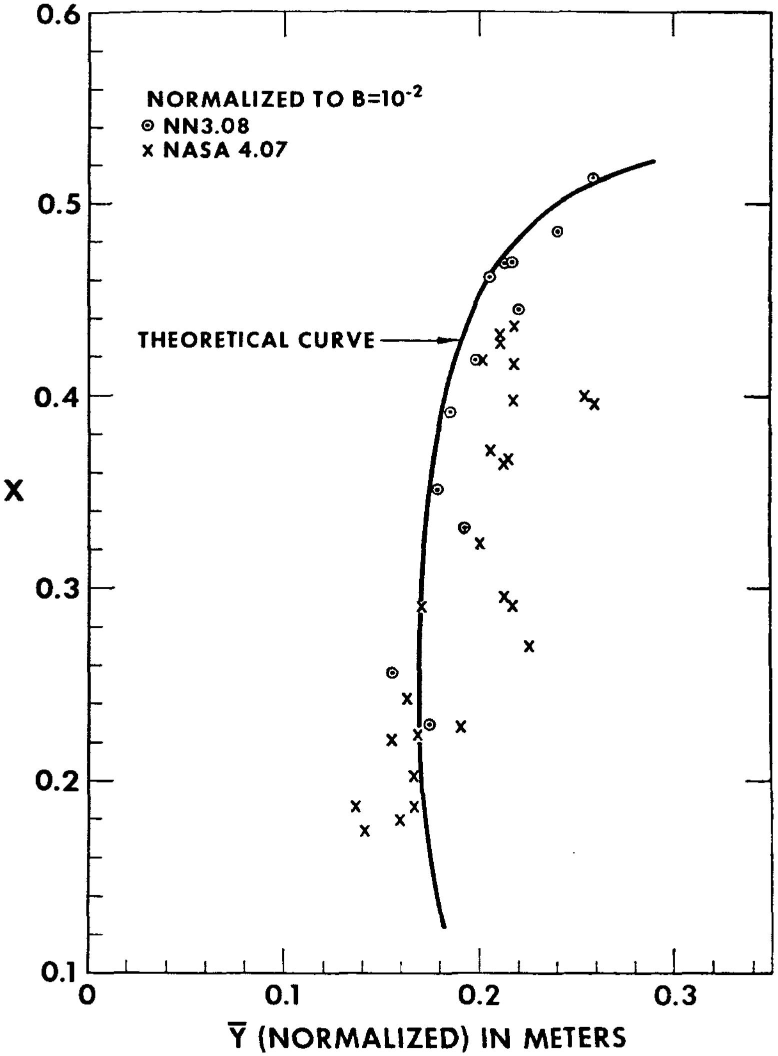

Fig. 3. Experimental measurements of sheath radius normalized to $B = 10^{-2}$.

voltages on the input and output sides of the antenna matching network. From these two measurements both the resistive and the reactive components of the antenna impedance can be derived [*Jackson and Kane*, 1959]. In the NASA 4.07 experiment, the approximate reactance was deduced only from a measurement at the output side of the antenna matching network; the matching circuit was fixed-tuned to resonate in free space, so that the output voltage was a measure of the detuning. Since no allowance for any resistive changes could be made in the second measurement, the results obtained can not be considered as accurate as those obtained from NN 3.08F. In both these flights the antennas were electrically short, so that their impedances were primarily capacitive. The reactance measurements then yielded the apparent dielectric constant K' ($= 1 - X'$) of the medium from the relationship

$$C' = K'C_0$$

where C' = measured capacitance, and C_0 = free space capacitance. From this experimental value of X' and a knowledge of the true value (obtained from a two-frequency propagation experiment) the size of a cylindrical region free of electrons around the antenna that would give the measured effective capacity was calculated in the manner indicated below.

Since the antenna is regarded as sufficiently long in comparison with the sheath radius to be considered infinitely long, the effect of the sheath can be calculated by considering the sheath edge as one plate of a cylindrical capacitor around the antenna. Then it is found that

$$X' = (Cs - C)/(C_s - XC)X \qquad (16)$$

where C_s = capacity of antenna to sheath per unit length, and C = free space capacity of antenna per unit length.

In the experimental measurements the values of X, X' are obtained. The value of C can be deduced from free space measurements of the antenna impedance (in these cases C was about 16 $\mu\mu$f/m for the NASA 4.07 rocket and about 19 $\mu\mu$f/m for the NN 3.08F rocket), so that C_s can be obtained from (16). The sheath radius is calculated from the capacity of this cylindrical condenser by equation 1. This radius was used (after normalizing to $B = 10^{-2}$) for the abscissa $\bar{y}$ in plotting the experimental results in Figure 3.

In the above treatment we have assumed that the sheath radius is constant along the length of the antenna; thus this approach is only applicable to electrically short antennas like those used on NN 3.08F and NASA 4.07, where the voltage is nearly constant along the length. Although the value of C, the free space capacitance of the antenna per unit length, is also variable because it depends, for example, on the spacing between the two halves of a dipole antenna, the value given for C can be regarded as effective average values.

For very long antennas the sheath radius is greatest at the voltage antinodes; these points are also the places where capacitance changes have most effect on the input impedance of the antenna. It has been found that applying expression (16) to a long antenna gives results in good agreement with experimentally measured impedances.

Under the conditions of the NN 3.08F rocket measurements, Table 1 is obtained for the ratio of the measured X' to the true X, and this ratio is compared to the ratio obtained from equation 16, using the theoretical value of C_s (obtained from $\bar{y}$) and using $C = 19$ $\mu\mu$f/m.

It is of interest that the theoretical value of the sheath radius for NN 3.08F (corresponding to $B = 5 \times 10^{-3}$) is about 13 cm, in good agreement with the value of about 6 inches (15.2 cm) published by *Jackson and Kane* [1959]. The column in Table 1 labeled 'Measured X'/X' can also be interpreted as N'/N, where N' is the apparent electron density and N is the true electron density. This ratio (or discrepancy) could not be explained previously, since the sheath thickness required appeared to be unreasonably large.

TABLE 1

Height km	Apparent X'	True X	Measured X'/X	Calculated X'/X
120	0.067	0.229	0.29	0.31
130	0.089	0.256	0.35	0.32
140	0.104	0.331	0.31	0.34
150	0.124	0.351	0.35	0.35
160	0.140	0.391	0.36	0.35
170	0.147	0.418	0.35	0.36
180	0.147	0.445	0.33	0.35
190	0.155	0.452	0.34	0.35
200	0.155	0.459	0.34	0.35
210	0.155	0.459	0.34	0.35
220	0.162	0.486	0.33	0.33
230	0.166	0.513	0.32	0.25

Conclusion

At frequencies above the plasma frequency, under conditions such that collisions and magnetic field effects can be neglected, it is possible to deduce, from the relatively simple analysis given above, the way in which an RF voltage applied to a cylindrical antenna affects the mean position of electrons close to the antenna. The results obtained are, within the limits of accuracy of the measurements that have been made, in good agreement with observations. The theory accounts for the discrepancies observed between true and apparent electron densities when a large RF voltage is applied to an RF impedance probe.

Note added in proof. Since the paper was written, it has come to my attention that a similar problem has been studied in connection with the containment of plasmas by RF fields. A review of the subject has been given by *Johnston* [1960].

Acknowledgments. The differential equation was solved by a Runge Kutta method on the Goddard Space Flight Center IBM 7090 computer. All the programming and computing were carried out by Robert F. Baxter of the Mathematics and Computing Branch.

This work was performed while I was a NASA senior postdoctoral research associate of the National Academy of Sciences.

REFERENCES

Bourdeau, R. E., J. E. Jackson, J. A. Kane, and G. P. Serbu, Ionospheric measurements using environmental sampling techniques, *Space Research, Proc. Intern. Space Sci. Symp., 1st, Nice, 1960,* edited by H. K. Kallmann-Bijl, pp. 328–339, North-Holland Publishing Company, Amsterdam, 1960.

Jackson, J. E., Rocket-borne instrumentation for ionosphere propagation experiments, *NRL Rept. 3909,* 1952.

Jackson, J. E., and J. A. Kane, Breakdown and detuning of transmitting antennas in the ionosphere, *NRL Rept. 5345,* 1959.

Jackson, J. E., and J. A. Kane, Performance of an RF impedance probe in the ionosphere, *J. Geophys. Res., 65,* 2209, 1960.

Jackson, J. E., and A. D. Pickar, Performance of a rocket-borne 7.75 Mc/s transmitting antenna in the ionosphere, *NRL Rept. 4940,* 1957.

Jastrow, R., and G. A. Pearse, Atmospheric drag on the satellite, *J. Geophys. Res., 62,* 413, 1957.

Johnston, T. W., *RCA Review, 21,* 570, 1960.

Kane, J. A., J. E. Jackson, and H. A. Whale, The simultaneous measurement of ionospheric electron densities by CW propagation and RF impedance probe techniques, *NASA Tech. Note D-1098,* 1962.

Rose, D. J., and Melville Clark, *Plasmas and Controlled Fusion,* p. 195, John Wiley & Sons, New York, 1961.

(Manuscript received August 21, 1962.)